

IEOR 251 Project: Competitive Facility Location

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May 14, 2004

Abstract

We intend to capture the competitive aspect of facility location in this report. In many real life facility location applications, the firm who locates the facility has to consider what benefit it gets, and how does the presence of other similar facilities affect the profit that this new facility earns. We present some models to analyze this situation. We present some ideas of competitive facility location on networks, and on a planar region, and show that one can find a dominant set of points that are sufficient to consider for finding the optimal locations of the facilities. We also consider two mixed integer programs for the discrete multifacility location problems and present lagrangean relaxation heuristics to solve them. We also compare the performance of the heuristics with the optimal solution given by a mixed integer program solver. Finally we present some ideas on solving a continuous multifacility location model using linear programming and formulate a mathematical program when all data are not known for certain (a robust MIP formulation).

1 Competitive Facility Location Model

The problem is to locate a new facility/facilities in the presence of other *competing facilities*. There is a finite set of customers or customers group denoted by I . Each customer $i \in I$ has a known location, and a strictly positive weight D_i , which is an indicator of the demand of the customer. The new facility has to compete with a finite number of old facilities. The set of the old facilities is denoted by R . An old facility $r \in R$ is located at site x_r and has a quality q_r , both of these are considered to be known.

Any customer $i \in I$ feels an attraction $attr(i, r)$ toward facility r at x_r , which depends on two factors i.e. the distance between the customer and the facility and the quality of the facility. Let the attraction function for the customer i be given by $A_i(q, d)$ which is a function of the quality q and the distance between the customer and the facility d . It is assumed that the function $A_i : [0, \infty] \times [0, \infty] \rightarrow [0, \infty]$ satisfies

- 1 The function is nondecreasing in the quality if the distance between the customer and the facility is fixed.
- 2 The function is non-increasing in the distance if the quality of the facility is fixed.

2 Examples of Attraction Functions

In general attraction functions can be of any type and may not be differentiable or even continuous. Some typical examples of attraction functions are

2.1 Gravity Type Attraction

The most common example of attraction function is of gravity type, which is given by:

$$A_i(q, d) = \frac{qk_i}{d^p} \quad (1)$$

Here p is some positive exponent, generally greater than 1 and k_i is some proportionality constant that may depend on specific customer i .

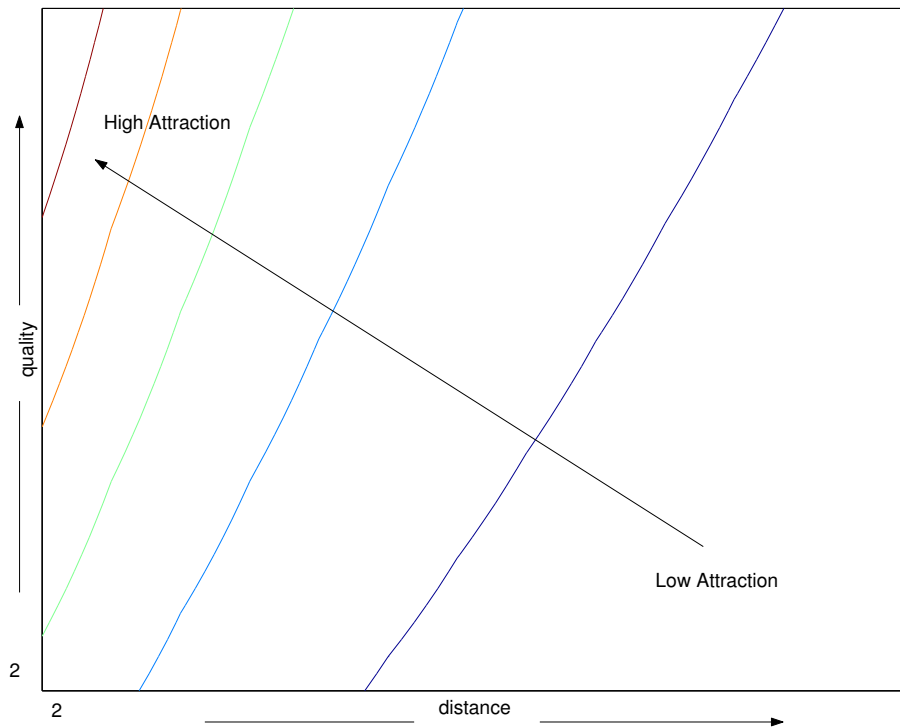


Figure 1: Iso-attraction curves for $A^{gravity}$ with $p = 2$

If $p = 2$, then the above model is equivalent to the famous gravitational law in Physics. This attraction function implies that the customer does not feel attracted toward the facility

very far away even though the quality may be high. The iso-attraction curves for this kind of model is shown in the figure.

One of the problem that arises when using these type of model is that the attraction approaches infinity as the distance approaches zero. If the distance of two facilities from a customer is zero then we have assumed that the customer will go the facility with higher quality level.

2.2 Additive Attraction Function

As mentioned in the section above, one of the problem when using the gravity type model as a measure of attraction is that the attraction approaches infinity as the distance approaches zero, and hence it is not possible for a customer to feel attracted toward a facility which is located elsewhere. But this does not make sense if there is a considerable difference in the qualities of the facilities. To overcome this difficulty we can model the attraction with an additive type of model which is given by:

$$A_i^{add}(q, d) = \max\{0, k_i q - c_i(d)\} \quad (2)$$

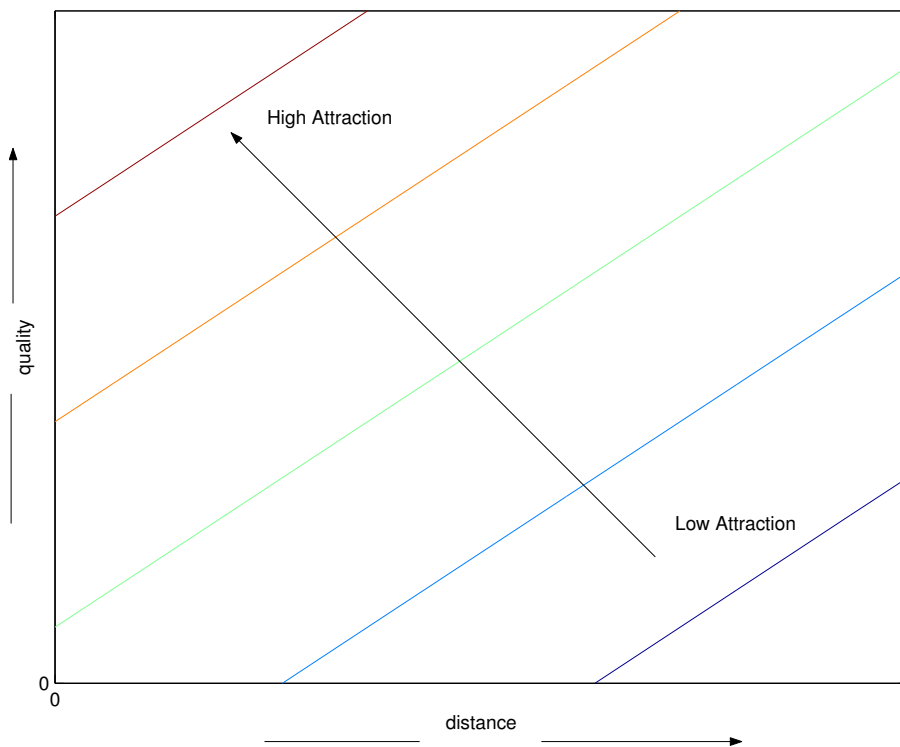


Figure 2: Iso-attraction curves for A^{add}

Here $k_i > 0$ and c_i is a nondecreasing function of d . An example of an attraction function is when $c_i(d) = c \cdot d$ where c is some constant of proportionality. Suppose k_i is the quantity that customer i need to buy and q is the price discount offered by the facility. Then this type of attraction function can be interpreted as total discount the customer will get minus the cost of transportation which is proportional to the distance traveled.

2.3 Non-continuous Attraction Function

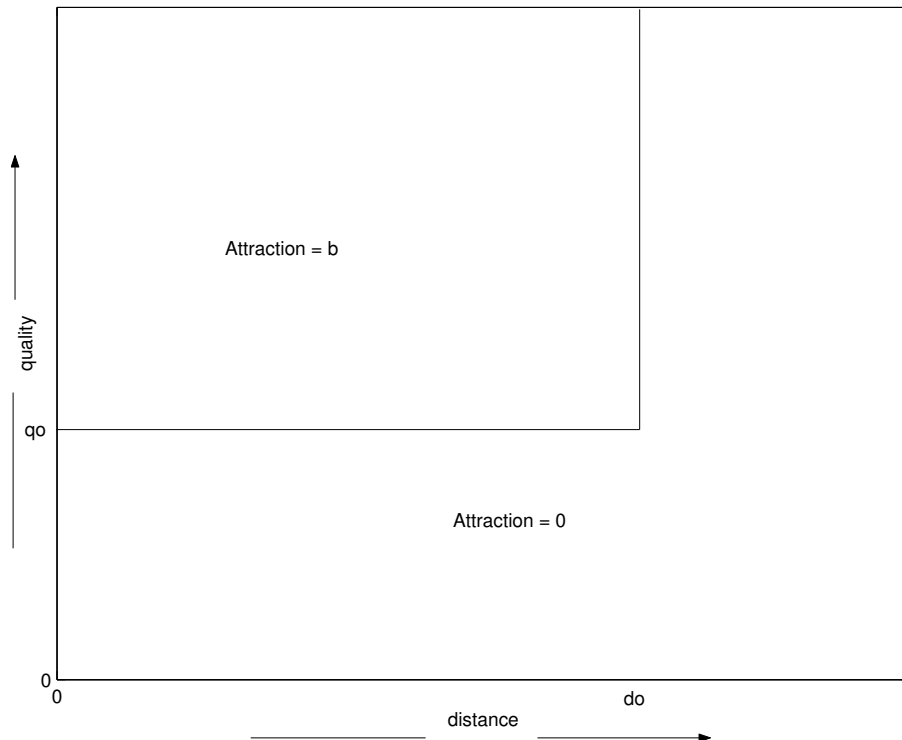


Figure 3: Iso-attraction curves for A^{step}

An example of the non-continuous attraction function is the step size function. Here attraction is only felt when the quality of the service is below some threshold ($q \geq q_0$) and the facility is not too distant ($d \leq d_0$). The attraction function is given as:

$$A_i^{step}(q, d) = \begin{cases} \beta_i & \text{if } q \geq q_0 \text{ and } d \leq d_0 \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

3 Competitive Facility Location on a Network

Suppose a network is given and the problem is to locate a maximum of B facilities on the network in presence of some existing facilities. Suppose a customer i is currently facing a

maximum attraction A_i to some facility. It is assumed that if the attraction of the new facility is greater than or equal to the maximum current attraction then the customer will buy all its quantity from the new facility. As the new facilities can be located anywhere on the network, and the quality of the service can take any arbitrary value in the interval $(0, \infty)$ there are infinite number of locations and quality pairs. However we can restrict our attention to a set of dominant (q, d) which satisfy some properties.

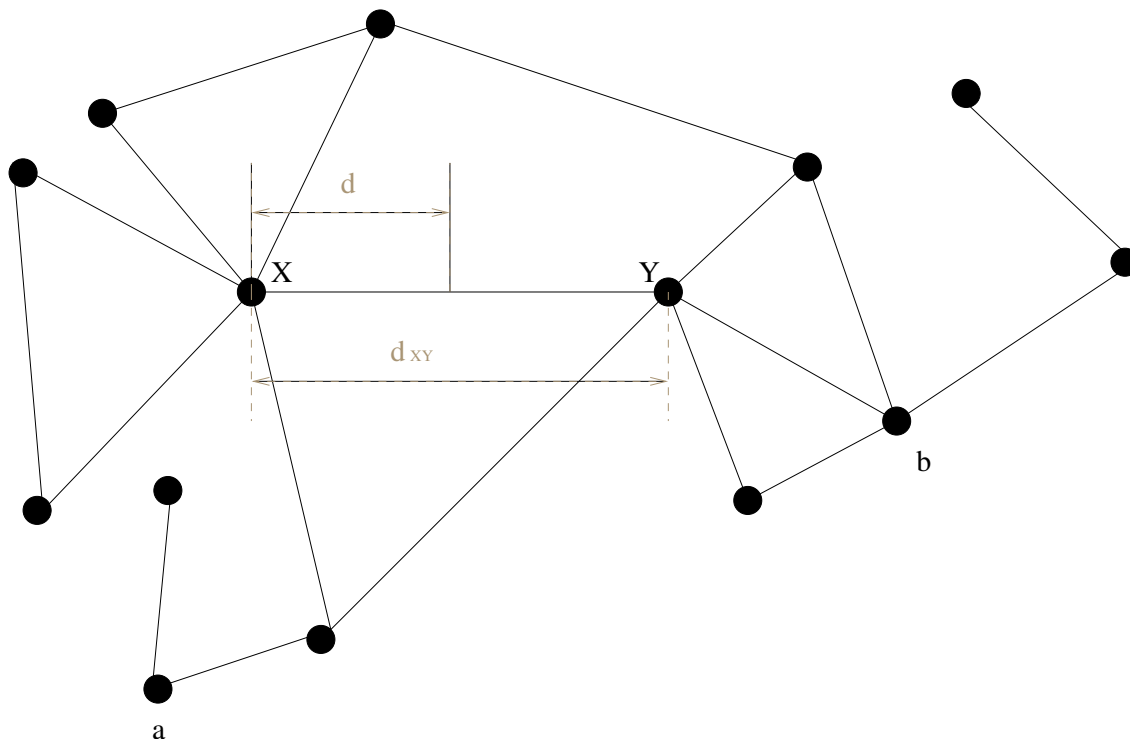


Figure 4: Competitive facility location on a network

Theorem 1 *If the attraction function is continuous and increasing in quality q and continuous and decreasing in the distance then in an optimum solution the point at which a new facility is located and the quality of the facility should be such that the attractions of it for at least two of the customers is equal to the current attractions to other facilities of these customers provided the location is not at a node itself.*

Proof Suppose we locate a facility on an edge between X and Y . For contradiction assume that there does not exist two customers for which the attraction to this facility is at their current attraction. If there exists no customer for which the attraction is at the current level then if we decrease the quality of the service by a small amount ϵ , then all customers that were previously attracted toward the facility remain attracted and hence we can decrease the cost by decreasing the quality. If there exists only one customer then we can move toward

the direction of that customer by a small amount and can also decrease the quality of the service by a small amount and still all the customers that were previously attracted remain attracted. So we can decrease the cost in this case too.

Moreover the point should be such that the shortest distance of one of those customer, for which the attraction to the new facility is same as current attraction, is through X and for the other customer the shortest distance should be through Y . It is easy to see that if that is not the case then we can still decrease the cost by moving on the edge toward both these customer.

Therefore the total points that need to be considered are all nodes and all points that satisfy the above properties.

3.1 Algorithm to find the Dominant Set of Points

- 1 Select an edge $X - Y$ of the network. Find the shortest path of all the nodes from X and from Y . Complexity of doing that is $O(n^2)$ where n is the total number of customers, if a simple version of Dijkstra's algorithm is used.
- 2 Divide the node in to two categories. If the shortest distance of a node from node X is smaller than that from node Y , put that node in the set S_x . Otherwise put the node in the set S_y .
- 3 Take a node from $a \in S_x$ having current attraction A_a . Take a node $b \in S_y$ having current attraction A_b . Let the shortest distance of node a from X is d_a and the shortest distance of the node b from Y is d_b . Let d_{XY} be the distance between X and Y . Solve the following two equations

$$A(q, d_a + d) = A_a \tag{4}$$

$$A(q, d_b + d_{XY} - d) = A_b \tag{5}$$

- 4 If the value of $q < \infty$ and the value of $d < d_{XY}$, then we have found a dominant (q, d) pair, otherwise discard the pair.
- 5 Repeat step [1]-[4] for every edge of the graph.
- 6 For every node (customer) of the network find the value of the quality that is required to attract this customer to the new facility considering the distance to be zero. If the value of $q < \infty$, then we have found a dominant pair, otherwise discard that pair.
- 7 Repeat this for every node in the network.

The complexity of the above algorithm is $O(mn^2L_1)$ where $O(L_1)$ is the complexity of solving the equations 4 and 5. If the network is a tree than the complexity of solving the problem reduces to $O(n^2L_1)$ because in the case of tree there exists a unique path between every pair of nodes.

4 Competitive Facility Location on a Plane

3-points uniquely define a plane. Therefore in the case of facility location in a plane the following equivalent theorem can be stated.

Theorem 2 *If the attraction function is continuous and increasing in quality q and continuous and decreasing in the distance then in an optimum solution the point at which a new facility is located and the quality of the facility should be such that the attractions of it for at least three of the customers is equal to the current attractions to other facilities of these customers provided that the facility is not on the line joining two customers. In that case at least two customers must be attracted at their current attraction level except the case when the facility is located at a distance of zero from some customer.*

Proof Proof is similar to the previous theorem. As the attraction function is continuous in distance and quality, we can always move the location or change the service level if the above condition does not hold.

If only two customers that are served from a facility are at their current attraction then the facility would always be located on the line joining these customers since otherwise we can decrease the cost by moving there (Provided triangle inequality holds).

4.1 Finding Dominant Set of Points

1. Pick three arbitrary customers i, j, k from the set of customers. Let their current attractions be A_i, A_j, A_k . Let the locations of these customers be given by $(a_i, b_i), (a_j, b_j), (a_k, b_k)$
2. Let the location and the quality of a new facility be (x, y) and q respectively. If this facility is such that the attractions of the facility for the above customers are equal to their current attractions, then it should satisfy

$$A(q, d_{(x,y)}(a_i, b_i)) = A_i \tag{6}$$

$$A(q, d_{(x,y)}(a_j, b_j)) = A_j \tag{7}$$

$$A(q, d_{(x,y)}(a_k, b_k)) = A_k \tag{8}$$

3. Consider all “well defined” (x, y, q) triplets that satisfies the above as potential solution.

4. Repeat step [1]-[3] for all possible combinations of i, j and k .

For two and one customer a procedure similar to finding a dominant set of points on a network can be used. The number of dominant triplets would be $O(n^3)$, and the complexity of finding all dominant points would be $O(n^3L_2)$, where $O(L_2)$ is the complexity of solving 6,7 & 8.

5 Finding the optimum solution

Once we have a dominant set of points with their respective service quality levels, we can formulate the following mixed integer program

$$\max \sum_i D_i y_i - \sum_j c q_j \quad (9)$$

subject to

$$A(q_j, d_{ij})x_j \geq u_{ij} \quad \forall i, j \quad (10)$$

$$y_i \geq u_{ij} \quad \forall i, j \quad (11)$$

$$y_i \leq \sum_j u_{ij} \quad \forall i \quad (12)$$

$$\sum_j x_j = B \quad (13)$$

$$x_j, y_i, u_{ij} \in \{0, 1\} \quad \forall i, j \quad (14)$$

This problem is similar to the discrete multi-facility location model. Hence we can say that the competitive multi-facility location problem on the plane is not much harder than the general multi-facility location model.

6 Integer programming formulations for Discrete multi-facility competitive location models

In this section we present some mixed integer programming formulations for the discrete multi-facility competitive location models and present some heuristics to solve the problems. We first consider two different formulations, one for multiplicative attraction function, and another for an additive attraction function. We present lagrangean relaxation heuristics to solve the problems.

6.1 Multiplicative attraction functions

6.1.1 The model

We have a set $I = \{1, \dots, m\}$ existing customers, each being attracted to some facility. The attraction of the customer i is A_i for the facility which currently serves customer i . We are not concerned about how many facilities (competitors) exist at this time, we assume that all relevant information is known and hence we can compute the attraction of a customer toward some facility and A_i denotes the maximum of all these attraction values. Hence, if we want to “capture” the demand for a particular customer $i \in I$, we need to provide an attraction of at least A_i to that customer.

We are allowed to locate up to a maximum of B facilities, and the potential locations for the facilities belong to the set $J = \{1, \dots, n\}$. If we provide a service level of q_j at a particular location $j \in J$, then customer i will be attracted to facility j with an attraction of value $k \frac{q_j}{d_{ij}}$, where k is a constant, and d_{ij} is the distance between facility j and customer i . Since we are in a discrete framework, the d_{ij} 's are all constants, and hence the attractions can be written as $f_{ij}q_j$. Let D_i denote the weight (demand) at each customer site i . The terms K_j denotes the fixed cost of opening a facility at location $j \in J$ and C_j denotes the variable cost of providing a unit service level at location j . Our objective is to locate some facilities so that our total revenue is maximized. For that purpose consider the following Mixed Integer program.

$$\text{Maximize } \sum_{i \in I} D_i u_i - \sum_{j \in J} K_j x_j - \sum_{j \in J} C_j q_j \quad (15)$$

$$\text{Subject to } q_j \leq Q x_j \quad \forall j \in J \quad (16)$$

$$\sum_{j \in J} x_j \leq B \quad (17)$$

$$f_{ij} q_j \geq A_i y_{ij} \quad \forall i \in I, j \in J \quad (18)$$

$$y_{ij} \leq u_i \quad \forall i \in I, j \in J \quad (19)$$

$$u_i \leq \sum_{j \in J} y_{ij} \quad \forall i \in I \quad (20)$$

$$x_j, y_{ij}, u_i \in \{0, 1\} \quad \forall i \in I, j \in J \quad (21)$$

$$q_j \geq 0 \quad \forall j \in J \quad (22)$$

The constraints can be interpreted in the following manner. Constraint 16 says that if I do not open a facility, there is no service level. Constraint 17 says that I can not locate more than B facilities. Constraint 18 says that customer i is captured by facility j only if it generates a greater attraction as compared to the existing server. Constraint 19 says that if one facility captures the demand of a customer, then I get the demand of that customer. Constraint 20 says that if none of my facilities capture a particular customer i , I do not get

anything from that customer.

We now present a lagrangean relaxation heuristic for the above MIP. We relax Constraints 19 and 20. The Dual problem for this case is

$$\min_{v_{ij}, w_i \leq 0} \max \sum_{i \in I} D_i u_i - \sum_{j \in J} K_j x_j - \sum_{j \in J} C_j q_j + \sum_{i,j} v_{ij} (y_{ij} - u_i) + \sum_i w_i (u_i - \sum_j y_{ij}) \quad (23)$$

subject to

$$q_j \leq Q x_j \quad \forall j \in J \quad (24)$$

$$\sum_{j \in J} x_j \leq B \quad (25)$$

$$f_{ij} q_j \geq A_i y_{ij} \quad \forall i \in I, j \in J \quad (26)$$

$$x_j, y_{ij}, u_i \in \{0, 1\} \quad \forall i \in I, j \in J \quad (27)$$

$$q_j \geq 0 \quad \forall j \in J \quad (28)$$

For given values of v_{ij} and u_i we can write the optimization problem as follows:

$$\max \sum_i (D_i + w_i - \sum_j v_{ij}) u_i - \sum_j K_j x_j - \sum_j C_j q_j + \sum_{i,j} (v_{ij} - w_i) y_{ij} \quad (29)$$

subject to

$$q_j \leq Q x_j \quad \forall j \in J \quad (30)$$

$$\sum_{j \in J} x_j \leq B \quad (31)$$

$$f_{ij} q_j \geq A_i y_{ij} \quad \forall i \in I, j \in J \quad (32)$$

$$x_j, y_{ij}, u_i \in \{0, 1\} \quad \forall i \in I, j \in J \quad (33)$$

$$q_j \geq 0 \quad \forall j \in J \quad (34)$$

Note that the u_i 's and the other variables are no longer linked in the problem. Hence the optimal solution to the above problem can be found as follows. To get the optimum values of the u_i 's, we can do the following procedure:

1. set $i \leftarrow 1$.
2. If $D_i + w_i - \sum_j v_{ij} > 0$ then set $u_i = 1$. Else set u_i to be zero.
3. $i \leftarrow i + 1$. Go to step 1.

Now we use the following procedure to find the optimum value of the other part. Note that for two different values of j , the problem can be separated. Hence the optimum can be found by evaluating the value of setting each x_j to 1 and then taking the best B of them. Here we show how to find the value of setting x_j to 1. Then, the maximum we can get is:

$$\max -K_j - C_j q_j + \sum_i (v_{ij} - w_i) y_{ij} \quad (35)$$

subject to

$$q_j \leq Q \quad (36)$$

$$f_{ij} q_j \geq A_i y_{ij} \quad \forall i \quad (37)$$

We can employ the following procedure to find the best we can do in this case:

1. Arrange the quantities $g_{ij} := \frac{A_i}{f_{ij}}$ in increasing order.
2. set $k = 1$.
3. Set $q_j = g_{kj}$ and evaluate the objective.
4. $k = k + 1$. Go to step 1.

The value of setting x_j to one can be found by taking the best of them all. Hence the best x_j 's can be found as follows:

1. For each x_j , find the value of setting it to one.
2. Take the best B x_j 's.

The above procedure gives us an upper bound for the optimum solution. Now we use the above solution to find a feasible solution that would give a lower bound on the optimum value. This can be done as follows. Given the values of x_j and q_j for all $j \in J$, we can determine the u_i 's as follows.

1. For each i
2. For each j
3. If $f_{ij} q_j \geq A_i$ for some j then set u_i to be 1.

6.1.2 Testing

In this section we outline the different testing experiments that were carried out to determine how good this lagrangean relaxation approach was. First of all we tested the model with $B = 1$, and with $K = C = A = 0$ to see whether the model gives the right answer to the single facility location problem. Once that was done, we tried to test our model with different data sets. The data sets were chosen so that one problem was of small size (5 customers) and the other problems contained 10 and 20 suppliers respectively. We wrote an AMPL code to solve the mixed integer program and then we tried to compare our results of the lagrangean relaxation with the MIP solution in terms of the following.

1. The difference in objective function values (as a percentage difference from MIP solution)
2. The number of new facilities located (as compared to the MIP solution)
3. The extent of demand captured by the new facilities (as a percentage of the MIP solution)

We now present the results for different cases:

The case of 5 customers

We experimented with B ranging from 1 to 5 with increasing maximum A_i for all the cases. The results are tabulated as shown. The objective function value for the Lagrangean Heuristic is taken as the lower bound at the end of iterations (since the upper bound solution need not be feasible).

	$A_i=1$	$A_i=2$	$A_i=3$	$A_i=4$	$A_i=5$
B = 1	100	100	100	100	100
B = 2	100	100	100	100	100
B = 3	100	97	100	100	100
B = 4	98	100	97	99	97
B = 5	100	94	97	96	92

Table 1: Objective value of lagrangean as a percentage of MIP objective

The next table shows the number of facilities located by the Lagrangean heuristic as compared to the number of facilities located by the MIP solver. Note that in some entries, even though the optimal objective values differ, the number of facilities located is the same. This is due to a suboptimal location of facilities by the Lagrangean heuristic.

	$A_i=1$	$A_i=2$	$A_i=3$	$A_i=4$	$A_i=5$
B = 1	0	0	0	0	0
B = 2	0	0	0	0	0
B = 3	0	0	0	0	0
B = 4	-1	0	0	0	1
B = 5	0	-1	1	-1	0

Table 2: No. of facilities located by heuristic - number of facilities in MIP.

We next compared the demand captured by the new facilities as a percentage of that of the MIP solution, and found that in all the above cases, the total demand captured by both the models were same.

The case of 10 Customers

We now present the comparisons of the results when we used 10 customers. Here too we varied B from 1 to 5 and A_i from 1 to 5. The results are shown in the following table. Each entry is a triplet (ob, nf, d) , where ob denotes the Lagrangean lowerbound objective as a fraction of the MIP objective, nf denotes the number of additional facilities located by the lagrangean heuristic, and d denotes the demand captured by the lagrangean model as a percentage of the demand captured by the MIP model.

	$A_i=1$	$A_i=2$	$A_i=3$	$A_i=4$	$A_i=5$
B = 1	100, 0, 100	100, 0, 100	100, 0, 100	100, 0, 100	100, 0, 100
B = 2	100, 0, 100	97, 0, 100	97, 0, 107	92, 0, 96	94, 0, 95
B = 3	100, 0, 100	94, 0, 95	94, 0, 94	90, 0, 87	91, 0, 95
B = 4	93, -1, 90	93, 0, 92	86, 0, 85	91, 0, 91	84, 1, 104
B = 5	92, 0, 89	87, -1, 90	84, 1, 104	90, 0, 91	86, 0, 95

Table 3: Results for the case of 10 suppliers

The case of 20 customers

The results are tabulated as shown. The meaning of each entry in the table is the same as that of the previous one.

We see that for a small number of customers, the lagrangean heuristic works quite well, but as the number of customers go up, it does not produce very good results. Some of the possible reasons could be that we relax more and more constraints as the problem grows

	$A_i=1$	$A_i=2$	$A_i=3$	$A_i=4$	$A_i=5$
B = 1	100, 0, 100	100, 0, 100	100, 0, 100	100, 0, 100	100, 0, 100
B = 2	89, 0, 100	87, 0, 100	91, 0, 100	82, 0, 90	80, 0, 88
B = 3	85, 0, 91	82, 0, 90	87, 0, 93	80, 0, 90	79, 0, 86
B = 4	81, 0, 87	78, 0, 86	79, 0, 85	75, 0, 81	81, 0, 100
B = 5	72, -1, 79	71, 0, 84	65, -1, 72	70, 1, 107	73, 0, 87

Table 4: Results for the case of 20 customers

bigger. Also, the number of constraints that we relax is $|I| + |I||J|$, which is a large number. We relaxed these many constraints so that the resulting problem can be solved easily, but then this is achieved at the quality of the solution. Hence it would be much better if we could come up with some other relaxation scheme that does not relax these many number of constraints. One could also try just relaxing Constraint 21, but then, the solution structure would not change much. Hence, we believe that this relaxation would also produce the same kind of result that we have obtained.

6.2 Additive attraction function

6.2.1 The Model

Now we consider the case of the additive attraction function, where the attraction of a particular customer i for a facility at location j is given by

$$attr(i, j) = k_1 q_j - k_2 d_{ij} \quad (38)$$

where k_1 and k_2 are constants. Hence, the MIP model for the additive model can be modified slightly to get the following formulation for the multiplicative model.

$$\text{Maximize } \sum_{i \in I} D_i u_i - \sum_{j \in J} K_j x_j - \sum_{j \in J} C_j q_j \quad (39)$$

$$\text{Subject to } q_j \leq Q x_j \quad \forall j \in J \quad (40)$$

$$\sum_{j \in J} x_j \leq B \quad (41)$$

$$k_1 q_j - k_2 d_{ij} x_j \geq A_i y_{ij} \quad \forall i \in I, j \in J \quad (42)$$

$$y_{ij} \leq u_i \quad \forall i \in I, j \in J \quad (43)$$

$$u_i \leq \sum_{j \in J} y_{ij} \quad \forall i \in I \quad (44)$$

$$x_j, y_{ij}, u_i \in \{0, 1\} \quad \forall i \in I, j \in J \quad (45)$$

$$q_j \geq 0 \quad \forall j \in J \quad (46)$$

Note that the above mathematical program is the same as that for the multiplicative model except for Constraint 42. Suppose we do not locate at location j . Then, the LHS of the constraint will be zero, which will force y_{ij} to be zero. If $y_{ij} = 1$ then we have that the LHS should be greater than A_i and then we will have that the attraction generated by the facility at location j will be greater than the current attraction level, and so the demand of customer i will be captured.

Note that we can again solve this problem by relaxing constraints 43 and 44 and then solve the relaxed problem for a specific value of v_{ij} and w_i , and then adjusting the values of the multipliers. Let us now consider (as in the multiplicative case) how to find the value of setting some x_j to one. As in the multiplicative case, we get the following mathematical program

$$\text{maximize} \quad -K_j - C_j q_j + \sum_i (v_{ij} - w_i) y_{ij} \quad (47)$$

$$\text{subject to} \quad q_j \leq Q \quad (48)$$

$$k_i q_j \geq A_i y_{ij} + k_2 d_{ij} \quad \forall i \in I \quad (49)$$

$$q_j \geq 0 \quad (50)$$

$$y_{ij} \in \{0, 1\} \quad \forall i \in I \quad (51)$$

This can be solved similar to the multiplicative model. The next section gives some idea about the results that we obtained using the lagrangean relaxation for the additive model.

6.2.2 Testing

We performed similar kind of testing for the additive attraction function model as we did in the multiplicative attraction function model. The results for the cases of 10 and 20 customers are as shown in the following tables. The notation is the same as that in the multiplicative attraction model.

	$A_i=1$	$A_i=2$	$A_i=3$	$A_i=4$	$A_i=5$
B = 1	100, 0, 100	100, 0, 100	100, 0, 100	100, 0, 100	100, 0, 100
B = 2	96, 0, 100	92, 0, 100	97, 0, 100	92, 0, 98	94, 0, 108
B = 3	95, 0, 100	96, 0, 100	90, 0, 104	91, 0, 100	93, 0, 96
B = 4	90, 0, 100	85, 0, 90	92, 0, 97	95, 0, 95	94, 1, 108
B = 5	84, -1, 86	80, 0, 84	76, -1, 80	70, 0, 87	79, 0, 92

Table 5: Results for the case of 10 customers

	$A_i=1$	$A_i=2$	$A_i=3$	$A_i=4$	$A_i=5$
B = 1	100, 0, 100	100, 0, 100	100, 0, 100	100, 0, 100	100, 0, 100
B = 2	87, 0, 100	90, 0, 100	86, 0, 100	84, 0, 95	79, 0, 84
B = 3	84, 0, 95	82, 0, 90	82, 0, 89	82, 0, 90	74, 0, 82
B = 4	81, 0, 89	76, 0, 92	71, 1, 104	78, 0, 80	78, 1, 108
B = 5	76, 0, 86	71, 0, 84	70, 0, 96	70, -1, 76	68, 0, 91

Table 6: Results for the case of 20 customers

In this case too we see that the lagrangean relaxation heuristic does not work well when the number of customers get larger.

7 Linear Attraction Function with Rectilinear Distances

Now a relatively simple model is considered which allows to formulate the problem as a mixed integer linear program. Even when the qualities of the services of the other facilities are unknown or known partially, a robust mixed integer linear program can be solved.

The attraction function for this model is given by

$$A_i(q, d) = k_1q - k_2d \quad (52)$$

Suppose J facilities need to be located in an area where there are R existing facilities. The various parameters and variables are

Parameters

- I Number of customers
- J Maximum number of new facilities
- a_i X-coordinate of the customer i
- b_i Y-coordinate of the customer i
- D_i Weight of the customer i
- c cost of increasing quality by one unit
- A_i Maximum attraction currently faced by customer i
- p Profit from a unit of demand captured

Variables

- x_j X-coordinate of the j^{th} new facility
- y_j Y-coordinate of the j^{th} new facility
- z_{ij} A binary variable denoting if the j^{th} facility cover the i^{th} customer
- q_j Quality of the j^{th} new facility

Let A be the maximum of all the attractions faced by any customer to any facility. Then a mathematical programming formulation for the above problem would be

$$Maximize \sum_{i=1}^I \sum_{j=1}^J pD_i z_{ij} - \sum_{j=1}^J cq_j$$

$$k_1 q_j - k_2(|x_j - a_i| + |y_j - b_i|) + A(1 - z_{ij}) \geq A_i \quad \forall i, j \quad (53)$$

$$\sum_{j=1}^J z_{ij} \leq 1 \quad \forall i \quad (54)$$

$$z_{ij} \in \{0, 1\} \quad (55)$$

The above formulation includes the absolute values and hence is not a linear program. We can, however, make it linear by making the following substitutions

$$x_j - a_i = p_{ij} - q_{ij} \quad |x_j - a_i| = p_{ij} + q_{ij} \quad (56)$$

$$y_j - b_i = r_{ij} - s_{ij} \quad |y_j - b_i| = r_{ij} + s_{ij} \quad (57)$$

where all $p_{ij}, q_{ij}, r_{ij}, s_{ij} \geq 0$. The new formulation is

$$Maximize \sum_{i=1}^I \sum_{j=1}^J pD_i z_{ij} - \sum_{j=1}^J cq_j$$

$$k_1 q_j - k_2(p_{ij} + q_{ij} + r_{ij} + s_{ij}) + A(1 - z_{ij}) \geq A_i \quad \forall i, j \quad (58)$$

$$\sum_{j=1}^J z_{ij} \leq 1 \quad \forall i \quad (59)$$

$$x_j - p_{ij} + q_{ij} = a_i \quad \forall i, j \quad (60)$$

$$y_j - r_{ij} + s_{ij} = b_i \quad \forall i, j \quad (61)$$

$$p_{ij}, q_{ij}, r_{ij}, s_{ij} \geq 0 \quad \forall i, j \quad (62)$$

$$z_{ij} \in \{0, 1\} \quad (63)$$

The above is a mixed integer program and can be solved using any IP solver. But most of the time the exact estimates of the qualities of the competing facilities are difficult to get. Moreover these values are subject to change over time. So there is a need to find a

robust solution to the above integer program which should work fine even when there is some change in the qualities. It is possible to have the qualities of different facilities to be correlated. For example if some of the competing facilities are owned by the same group then we expect some positive correlation among them. The correlation can be negative too if we consider there is some budget constraint on the total amount of money that can be spent on maintaining the qualities.

Let R be the total number of existing facility and v_r denotes the quality of the r^{th} facility. b_{ir} denotes the distance between i^{th} customer and r^{th} facility. Suppose the value of v_r is unknown but it is supposed to lie in the interval $\{\bar{v}_r - v_r^0, \bar{v}_r + v_r^0\}$. Let $t_r = \frac{v_r - \bar{v}_r}{v_r^0}$. the value of each z_r lies between -1 and 1 . Suppose in addition we have an $L \times R$ matrix Q such that $Qv = h$, where h is an L dimensional vector. The purpose of the matrix Q is to capture the correlation among the qualities. In addition we have a restriction

$$\sum_{r=1}^R |t_r| \leq \Gamma \quad (64)$$

The above constraint restricts the sum of deviation in the values of t_r . As it is very unlikely that all the t_r 's are at value 1 or -1 at the same time. If we let $\Gamma = R$, then it is equivalent to doing the worst case scenario which is extremely unlikely. If we let $\Gamma = 0$, then it is equivalent to the MIP stated above. Let $V = \{v : v_r = \bar{v}_r + t_r v_r^0 \forall r, Qv = h, \sum_{r=1}^R |t_r| = \Gamma\}$.

With these restrictions the mathematical program is

$$\max_{z,q} \min_{v \in V} \sum_{i=1}^I \sum_{j=1}^J p D_i z_{ij} - \sum_{j=1}^J c q_j$$

$$k_1 v_r - k_2 b_{ir} \leq A_i \quad \forall i, r \quad (65)$$

$$k_1 q_j - k_2 (p_{ij} + q_{ij} + r_{ij} + s_{ij}) + A(1 - z_{ij}) \geq A_i \quad \forall i, j \quad (66)$$

$$\sum_{j=1}^J z_{ij} \leq 1 \quad \forall i \quad (67)$$

$$x_j - p_{ij} + q_{ij} = a_i \quad \forall i, j \quad (68)$$

$$y_j - r_{ij} + s_{ij} = b_i \quad \forall i, j \quad (69)$$

$$p_{ij}, q_{ij}, r_{ij}, s_{ij} \geq 0 \quad \forall i, j \quad (70)$$

$$z_{ij} \in \{0, 1\} \quad (71)$$

This kind of robust formulations are considered in [2] and can be solved as a mixed integer program by taking the dual of the inner minimization problem.

8 Conclusion

We have shown that the competitive multi-facility location problem on a network or on a plane is not much harder than the general multi-facility location problem if the attraction function is continuous and non-increasing in distance and continuous and non-decreasing in quality or service level. Our results do not depend on a specific type of attraction function or distance norm. For general attraction function it require solving non-linear equations in 2 or 3 variables, complexity of solving these may depend on the specific attraction function used. We have also considered discrete competitive multi-facility location problems and presented a lagrangean based heuristics for solving those problems. In the end we formulated a robust mixed integer program when the distances are rectilinear and the attraction is additive.

References

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